Acyclic Colorings of Weakly Chordal Graphs

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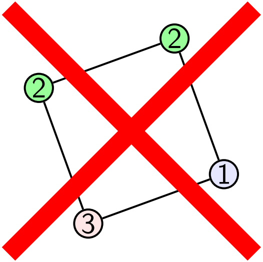
Background Concepts

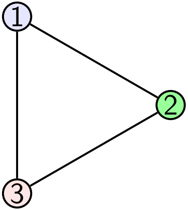
- Graph Coloring
- Chordal (triangulated) graphs
- Triangulations (chordal completions)
- Weakly chordal graphs
- Treewidth

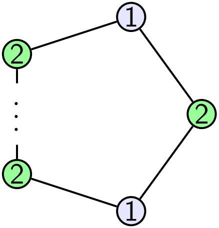
Outline

Coloring

Proper vertex coloring



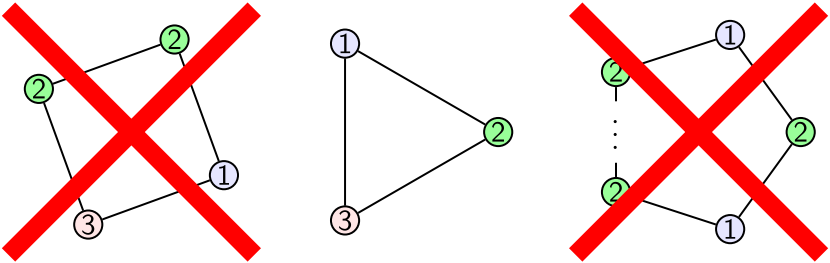




chromatic number $\chi(G)$

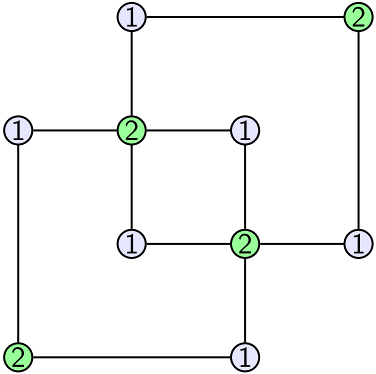
Acyclic Coloring

Proper vertex coloring without bichromatic cycles

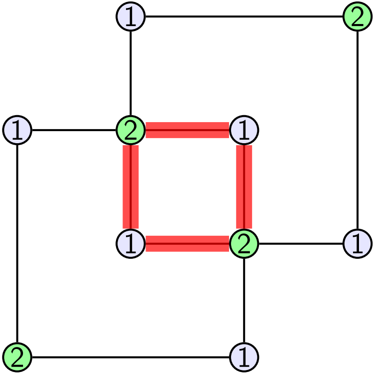


acyclic chromatic number $\chi_a(G) \geq \chi(G)$

Acyclic Coloring – No Bichromatic Cycles



Acyclic Coloring – No Bichromatic Cycles



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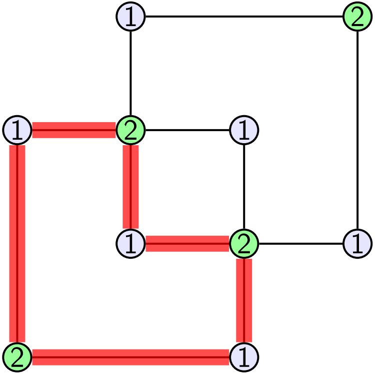
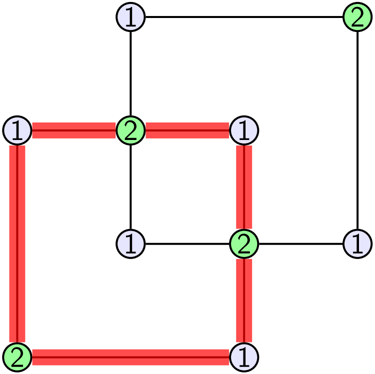


Acyclic Coloring – No Bichromatic Cycles



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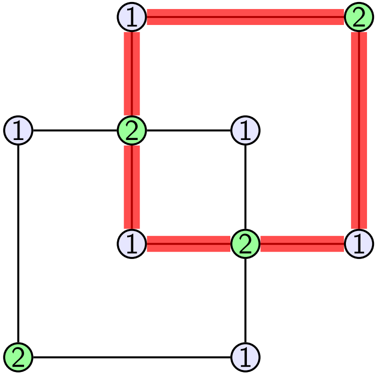

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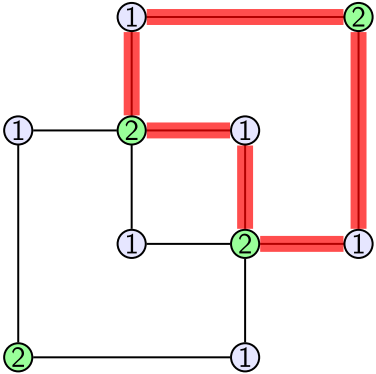

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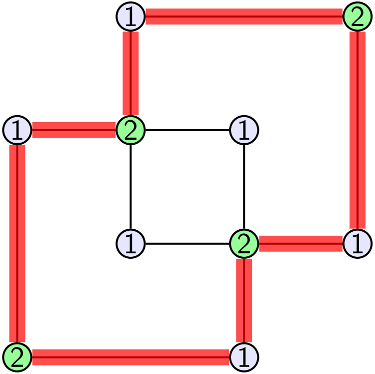
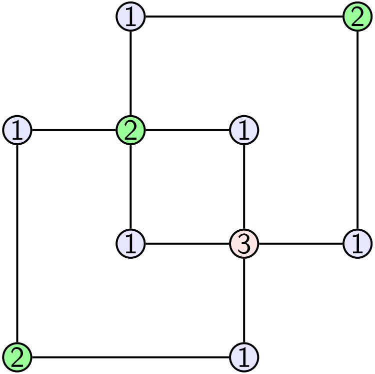
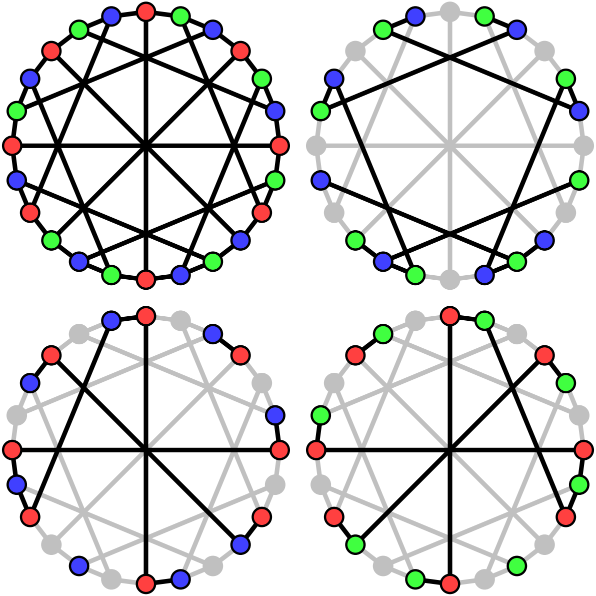


Acyclic Coloring – No Bichromatic Cycles



$$\chi_a(G) = 3$$



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http://commons.wikimedia.org/wiki/File:Acyclic_coloring.svg

ACYCLIC COLORINGS OF PLANAR GRAPHS†

BY

BRANKO GRÜNBAUM

ABSTRACT

A coloring of the vertices of a graph by k colors is called acyclic provided that no circuit is bichromatic. We prove that every planar graph has an acyclic coloring with nine colors, and conjecture that five colors are sufficient. Other results on related types of colorings are also obtained; some of them generalize known facts about "point-arboricity".

1. Introduction

Let G denote a graph with vertex set V; we shall assume th a G contains no 1- or 2-circuits (that is, loops or multiple edges). A *k-coloring* of G is a partition $V = V_1 \cup \cdots \cup V_k$ of the vertices of G into k pairwise disjoint sets (called *colors*) so that adjacent vertices are in different sets (have different colors). A k-coloring of G is called *acyclic* provided that every subgraph of G spanned by vertices of two of the colors is acyclic (in other words, is a forest). If G is the graph of the octahedron then the 4-coloring of G indicated in Fig. 1 by the numerals placed near the vertices is not acyclic (since the colors 1 and 2 span a graph which is not

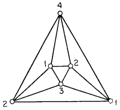

Fig. 1.

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ON ACYCLIC COLORINGS OF PLANAR GRAPHS

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The conjecture of B. Grünbaum on existing of admissible vertex coloring of every planar graph with 5 colors, in which every bichromatic subgraph is acyclic, is proved and some corollaries of this result are discussed in the present paper.

1. Introduction and statement of the result

In 1973 Grünbaum has published a large paper [5] on graph colorings, in which various restrictions were given to the type of all 2- and 3-chromatic subgraphs. The main attention in this paper was attached to the planar graphs.

Definition 1. An admissible coloring of a graph is called *acyclic* (in narrow sense), if every bichromatic subgraph, induced by this coloring, is a forest (acyclic graph).

The acyclic coloring of a graph should obviously be considered only for loopless graphs without multiple edges, which is assumed below.

The first example of a planar graph, which is not acyclically 4-colorable, has been constructed by Grünbaum [5]. Afterwards Wegner has constructed [12] a planar graph, which possess a cycle in every 2-chromatic subgraph in every admissible 4-coloring.

Definition 2. Graph G is called *k-degenerated*, if each subgraph H of G contains a vertex, which induced degree is less than k, i.e.

$$W(G) = \max_{G' \subseteq G} \min_{v \in V(G')} s_{G'}(v) + 1 \leq k,$$

where $W(G)$ is known as Vizing-Wilf's number.

In particular, a graph is 1-degenerated, iff it contains no edges, and is 2-degenerated, iff it is a forest.

Kostochka and Melnikov have shown [8] (answering Grünbaum's question), that graphs, acyclically not colorable with 4 colors, can be found even among 3-degenerated bipartite planar graphs.

Acyclic Coloring, Algorithmically

Acyclic Coloring (AC)
Instance: Graph G, positive integer k.
Question: Is there an acyclic coloring of G that uses $\leq k$ colors?

NP-Complete to determine whether $\chi_a(G) \leq 3$ (Kostochka 1978)

If $\Delta(G) \leq 3$, then G can be acyclically colored using 4 colors or fewer in linear time. (Skulrattanakulchai 2004)
If $\Delta(G) \leq 5$, then G can be acyclically colored using 9 colors or fewer in linear time. (Fertin & Raspaud 2008)

ACYCLIC COLORING, Algorithmically

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THE CYCLIC COLORING PROBLEM AND ESTIMATION OF SPARSE HESSIAN MATRICES*

THOMAS F. COLEMAN† AND JIN-YI CAI†

Abstract. Numerical optimization algorithms often require the (symmetric) matrix of second derivatives, $\nabla^2 f(x)$. If the Hessian matrix is large and sparse, then estimation by finite differences can be quite attractive since several schemes allow for estimation in much fewer than n gradient evaluations.

The purpose of this paper is to analyze, from a combinatorial point of view, a class of methods known as substitution methods. We present a concise characterization of such methods in graph-theoretic terms. Using this characterization, we develop a complexity analysis of the general problem and derive a roundoff error bound on the Hessian approximation. Moreover, the graph model immediately reveals procedures to effect the substitution process optimally (i.e. using fewest possible substitutions given the differencing directions) in space proportional to the number of nonzeros in the Hessian matrix.

Key words. graph coloring, estimation of Hessian matrices, sparsity, differentiation, numerical differences, NP-complete problems, unconstrained minimization

AMS(MOS) subject classifications. 65K05, 65K10, 65H10, 68L10

1. Introduction. We are concerned with the estimation of a large sparse symmetric matrix of second derivatives $\nabla^2 f(x)$ for some problem function $f: R^n \to R^1$. In particular, we note that the product $\nabla^2 f(x) \cdot d$ can be estimated, for example, by forward differences

$$\nabla^2 f(x) \cdot d = [\nabla f(x+d) - \nabla f(x)] + o(\|d\|). \tag{1.1}$$

When the structure of $\nabla^2 f(x)$ is known, then usually a few well chosen differencing directions $d_1, \cdots, d_p$ affords the recovery of estimates of all nonzeros of $\nabla^2 f(x)$. Let us denote our estimate by H. We will assume that the sparsity pattern of H is known; the diagonal elements are specified as nonzero; H is symmetric. (Restricting the diagonal to be zero-free is reasonable in many contexts: In particular, a minimizer of f usually possesses a positive definite Hessian matrix.) We will be concerned with

↑ NP-complete even when restricted to bipartite graphs

Acyclic Coloring, Algorithmically

Chordal Graphs

- Every proper coloring is also an acyclic coloring (in particular, $\chi_a(G) = \chi(G) = \omega(G)$). (Bodlaender et al. 2000, Gebremedhin et al. 2009)
- Chordal graphs can be colored in $O(n + m)$ time.

Acyclic Coloring, Algorithmically

Chordal Graphs

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Cographs (L. 2009)

Also known as the P_4-free graphs

- The cographs are *exactly* the graphs for which every acyclic coloring is also a star coloring.
- An optimal acyclic (and star) coloring of a cograph can be found in $O(n)$ time (if a cotree is given as part of the input).

Triangulating Colored Graphs

Definition (ϕ-triangulatable)

Let ϕ be a proper coloring of a graph G. G is ϕ-triangulatable if there exists a triangulation H of G such that ϕ is a proper coloring of H.

Triangulating Colored Graphs, Algorithmically

Triangulating Colored Graphs (TCG)
Instance: Graph G and a proper coloring ϕ of G.
Question: Is G ϕ-triangulatable?

This problem is hard (Bodlaender et al. 1992). TCG is...

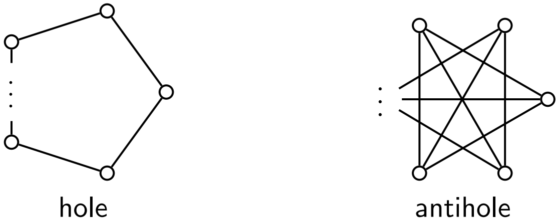
...NP-complete even when each color class has exactly two vertices.

...$W[t]$-hard for all $t \in \mathbb{N}$.

Weakly Chordal Graphs

Definition

A graph is weakly chordal if it contains no induced hole or antihole on five or more vertices.



Forbidden induced subgraphs for weakly chordal graphs.

Weakly Chordal Graphs

Lemma

If ϕ is a proper coloring of a weakly chordal graph G, then an edge $uv \in E(G)$ is contained in a bichromatic cycle if and only if uv is contained in a bichromatic C_4.

Outline

The Main Result

Theorem

If ϕ is a proper coloring of a weakly chordal graph G, then ϕ is an acyclic coloring of G if and only if ϕ is a proper coloring of some triangulation of G.

Proof.

("If"): Trivial.

("Only if"): Show that G can be triangulated without creating a bichromatic cycle. $\square$

Tools – Two-Pairs

Definition (Two-pair)

A pair $\{x, y\}$ of distinct, non-adjacent vertices is a two-pair if every induced path from x to y consists of exactly two edges.

Theorem (Hayward, Hoàng, and Maffray 1989)

If G is a weakly triangulated graph, then every induced subgraph of G that is not a clique contains a two-pair.

Tools – Separators

Definition

Let G be a connected graph. $S \subset V(G)$ is a...

- **separator** *if $G - S$ is disconnected.*
- **x-y-separator** *if x and y are contained in distinct components of $G - S$.*
- **clique separator** *if $G[S]$ is a clique.*

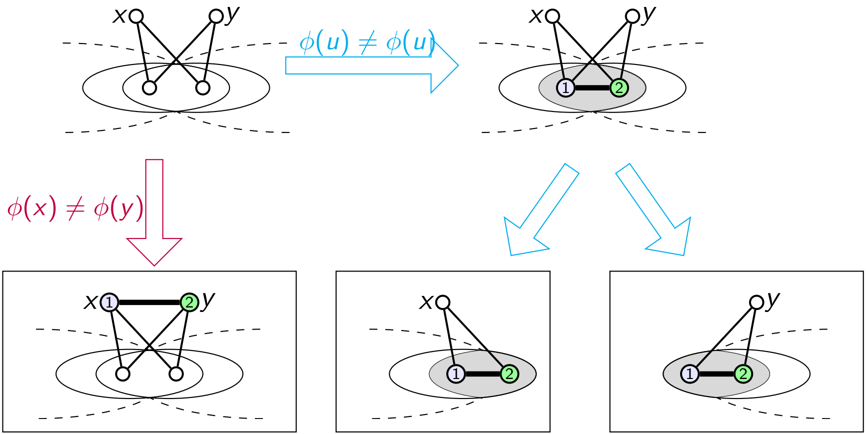
Lemma

Let G be a graph with clique separator S and let ϕ be a proper coloring of G. G is ϕ-triangulatable if and only if $G[S \cup R]$ is ϕ-triangulatable for every connected component R of $G - S$.

The Key Lemma

If ϕ is an acyclic coloring of a graph G with two-pair $\{x, y\}$, then either $\phi(x) \neq \phi(y)$ or $\phi(u) \neq \phi(v)$ for all $u, v \in N(x) \cap N(y)$.

Proof.



Connecting a Two-Pair

Lemma (Spinrad and Sritharan, 1995)

If $\{x, y\}$ is a two-pair in a graph G, then G is weakly chordal if and only if $G + xy$ is weakly chordal.

Lemma

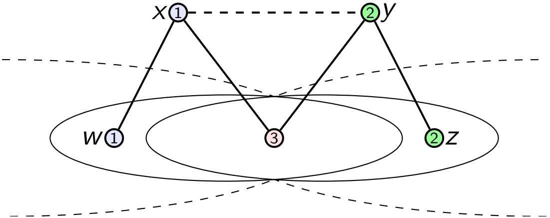
Let ϕ be an acyclic coloring of a graph G. If $\{x, y\}$ is a two-pair in G such that $\phi(x) \neq \phi(y)$, then ϕ is an acyclic coloring of $G + xy$.

Connecting a Two-Pair

Lemma

Let ϕ be an acyclic coloring of a graph G. If $\{x, y\}$ is a two-pair in G such that $\phi(x) \neq \phi(y)$, then ϕ is an acyclic coloring of $G + xy$.

Proof.



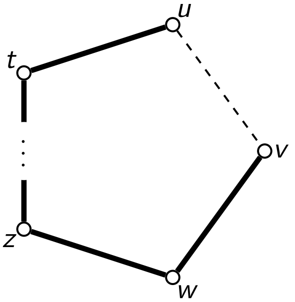
□

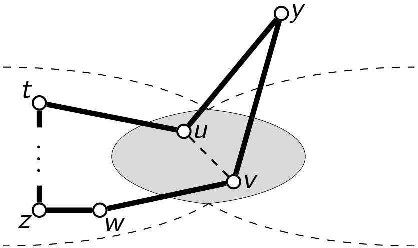
Completing $N(x) \cap N(y)$

Lemma

If $\{x, y\}$ is a two-pair in a weakly chordal graph G, then the graph obtained by turning $N(x) \cap N(y)$ into a clique is weakly chordal.

Proof.





The addition of edge uv cannot create a hole.

□

Completing $N(x) \cap N(y)$

Lemma

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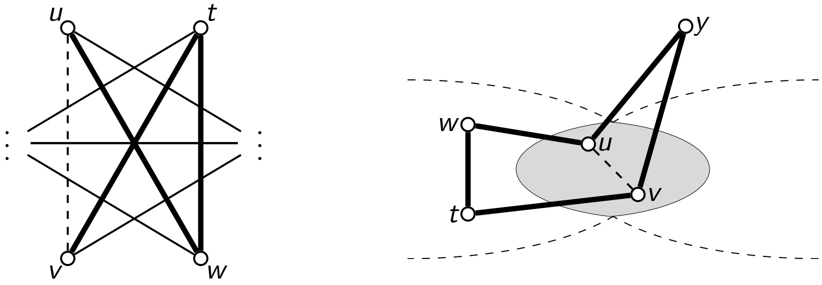
Proof.



The addition of edge uv cannot create a antihole.

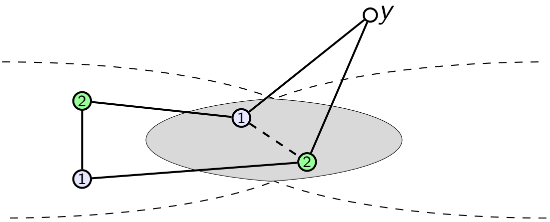
□

Completing $N(x) \cap N(y)$

Lemma

Let $\{x, y\}$ be a two-pair in a weakly chordal graph G and let $S = N(x) \cap N(y)$. If ϕ is an acyclic coloring of G such that $\phi(u) \neq \phi(v)$ for all $u, v \in S$, then ϕ is an acyclic coloring of G_S.

Proof.



We've shown that adding edge uv cannot create a hole or an antihole; we still need to show that it cannot create a bichromatic C_4. □

Outline

Algorithms

Our algorithm (our proof, really) is useless

TRIANGULATING COLORED GRAPHS
: All we have to do is check whether ϕ is an acyclic coloring. This can be done in polynomial (linear?) time.

TRIANGULATING COLORED GRAPHS
Just as simple, but not as obvious...

Treewidth

Definition (Treewidth)

The ***treewidth*** $\mathrm{tw}(G)$ *of a graph* G *is*

$$\min\{\omega(H) \mid H \text{ is a triangulation of } G\} - 1.$$

Theorem (Bouchitté and Todinca 1999)

Treewidth *can be solved in polynomial time* $O(n^6)$ *on weakly chordal graphs.*

Treewidth

Definition (Treewidth)

The ***treewidth*** $\mathrm{tw}(G)$ *of a graph* G *is*

$$\min\{\omega(H) \mid H \text{ is a triangulation of } G\} - 1.$$

The Key: chordal graphs are perfect!

Theorem (Bouchitté and Todinca 1999)

Treewidth *can be solved in polynomial time* $O(n^6)$ *on weakly chordal graphs.*

Corollary

Every weakly chordal graph G *satisfies* $\chi_a(G) = \mathrm{tw}(G)$.

Corollary

Acyclic Coloring *can be solved in polynomial time on weakly chordal graphs.*

Constructive Algorithms

Note: given tw-optimal triangulation we can find an optimal acyclic coloring in $O(n + m)$ time (this is coloring chordal graphs).

Theorem

If $\mathcal{C}$ is a subclass of the weakly chordal graphs for which TREEWIDTH *can be solved constructively in $f_{\mathcal{C}}(n, m)$ time for every $G \in \mathcal{C}$, then an optimal acyclic coloring can be constructed in $O(f_{\mathcal{C}}(n, m) + n + m)$ time for every $G \in \mathcal{C}$.*

Outline

Open Problems

Open Problems

- Characterize the graphs for which ϕ acyclic $\Leftrightarrow$ G is ϕ-triangulatable.
- Can we beat the best known algorithm for treewidth (and thus acyclic coloring) on weakly chordal graphs?
- $O(n)$ time constructive algorithm for $(q, q-4)$ graphs
- $O(n)$ time constructive algorithm for distance-hereditary graphs
- Can Acyclic Coloring be solved in polynomial time on graphs with bounded treewidth?

Thank You!

Questions?